acceleration becomes

$$M = x + t_g v + \tau^2(e^{-T} + T - 1)a_m - (t_g^2/2)a_{T_n} \quad (16)$$

Air-to-Ground Interception

The optimal closed loop guidance law given by Eqs. (14) and (16) is rewritten as

$$u = (\Lambda/t_g^2)(x + t_g v) + (\Lambda/T^2) \times (e^{-T} + T - 1)a_m - (\Lambda/2)a_{T_n} \quad (17)$$

It is desirable to express the state variables x and v in terms of parameters most commonly employed in tactical missile guidance design. If it is assumed that the reference intercept course is at any instant aligned with the line-of-sight, then

$$x = 0, \quad v = -R\dot{\sigma} \quad (18)$$

With the time-to-go approximated as

$$t_g = -R/\dot{R} \quad (19)$$

the optimal guidance law can be expressed as

$$u = \Lambda\dot{R}\dot{\sigma} + (\Lambda/T^2)(e^{-T} + T - 1)a_m - (\Lambda/2)a_{T_n} \quad (20)$$

Equation (20) is a biased proportional navigation guidance law where a time varying navigation gain and an acceleration feedback path provide compensation for the missile time lag. The target acceleration is the component of gravity normal to the line-of-sight. Figure 2 displays a block diagram representation of the air-to-ground guidance law.

Air-to-Air Interception

The quantity $[(\Lambda/2)a_m]$ is added to and subtracted from Eq. (17) to obtain

$$u = (\Lambda/t_g^2)[x + t_g v + (t_g^2/2)a] + (\Lambda/T^2)(e^{-T} + T - 1)a_m - (\Lambda/2)a_m \quad (21)$$

where

$$a = a_m - a_{T_n} \quad (22)$$

For the case where the reference intercept course is defined as the instantaneous line-of-sight, Eq. (22) may be written as

$$a = -2\dot{R}\dot{\sigma} - R\ddot{\sigma} \quad (23)$$

and the acceleration command normal to the line-of-sight becomes

$$u = (\Lambda\dot{R}t_g/2)\ddot{\sigma} + (\Lambda/T^2)(e^{-T} + T - 1)a_m - (\Lambda/2)a_m \quad (24)$$

A block diagram of the air-to-air guidance law is shown in Fig. 3.

Comments on the Optimal Guidance Laws

One of the most successfully employed tactical missile guidance techniques is proportional navigation. Trajectory simulations of simplified missile-target dynamics have shown that the terminal miss distance is substantially reduced for missiles employing the optimal guidance laws compared to miss distances resulting from proportional navigation. The air-to-ground optimal law has been incorporated into a high-order nonlinear missile dynamics simulation which yielded equally substantial performance improvements in terms of reduced miss distances and the infrequent occurrence of acceleration command saturation. The practicality of considering higher order missile dynamics in the guidance law synthesis is questionable when hardware implementation is involved. Each additional state variable defined in the differential equations of constraint appears in the guidance law as a time varying feedback term. The implication is that significant miss distance reductions are attainable through the optimal guidance laws when high-order missile dynamics are represented by a single time lag. Time-to-go estimations also introduce implementation problems. For missiles equipped with infrared and electrooptical sensing devices, passive time-to-go estimations such as those described by Rawling[5] are most promising.

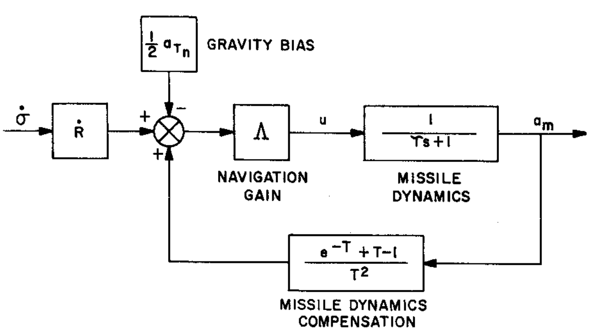

Fig. 2 Short-range air-to-ground guidance law.

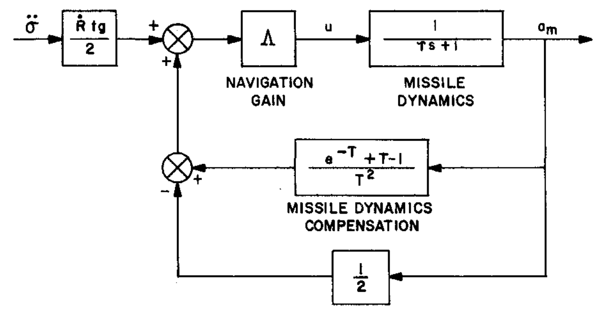

Fig. 3 Short-range air-to-air guidance law.

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Recovery Factor for Highly Accelerated Laminar Boundary Layers

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Nomenclature

C = $\rho\mu/\rho_e\mu_e$
E = $u_e^2/2H_e$, Mach number, or flow kinetic energy, parameter
f = dimensionless stream function
g = H/H_e

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H,h = total, and static enthalpy, respectively
Pr = Prandtl number
R = radius
r = $1 - (1 - g_0)/E$, recovery factor
s,y = streamwise and normal coordinates, respectively
u = streamwise velocity component
β = $2(1 - E)^{-1} dlnu_e/dln\xi$, acceleration parameter
ϵ = geometrical index
η = $[\rho_e u_e/(2\xi)^{1/2}] \int_0^y R^\epsilon (\rho/\rho_e) dy$, transformed normal coordinate
μ = dynamic viscosity
ξ = $\int_0^s \rho_e u_e \mu_e R^{2\epsilon} ds$, transformed streamwise coordinate
ρ = density
ϕ = function defined by Eq. (6)
ω = exponent in viscosity enthalpy relation, $\mu \alpha h^\omega$

Subscripts

e = freestream value
0 = adiabatic surface value

THE purpose of this Note is to resolve a controversy existing in the literature concerning the recovery factor for highly accelerated laminar boundary-layer flows. Numerical data have not been heretofore available owing to the inadequacy of generally used calculation procedures for boundary-layer flows. Dewey and Gross[1] suggest that $r = 1.0$ in the limit $\beta = \infty$ for all values of Pr, E and ω. On the other hand, Back[2] recommends that $r = Pr^{1/2}$ should be used in heat-transfer calculations for $0 \leq \beta \leq 20$. Presented here are results of computations of adiabatic compressible laminar boundary layers for $0 \leq \beta \leq 20$. The recovery factor is found to be a monotonically decreasing function of β, the value $r \simeq Pr^{1/2}$ obtaining only in the vicinity of $\beta = 0$.

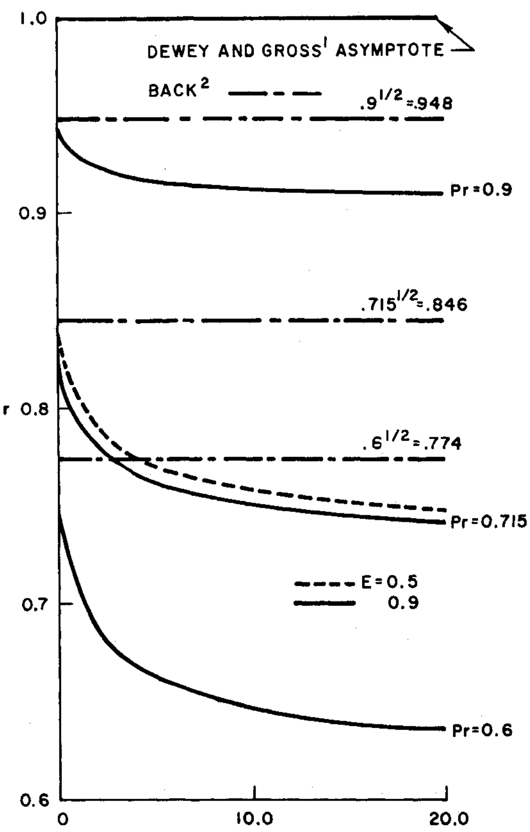

Fig. 1 Recovery factor as a function of the acceleration parameter; ($\mu \alpha h^{0.5}$). Comparison with recommendations of Dewey and Gross[1] and Back.[2]

Furthermore, the data indicate an asymptote of $r \rightarrow Pr$ as $\beta \rightarrow \infty$; we present an analysis to confirm this observation.

The governing equations for self-similar adiabatic compressible laminar boundary layers were presented in Ref. 3; the method of solution and accuracy considerations are discussed in detail in Refs. 4 and 5. In order to show clearly the influence of Prandtl number, model gases with three different but constant values of Pr (0.6, 0.715, 0.9) were considered. Both the effects of the exponent ω in the viscosity-enthalpy relation, and the Mach number were extensively studied for $\beta \leq 2$ in Refs. 3 and 5. Thus for the present purpose it is sufficient to consider only $\omega = 0.5$, and demonstrate the effect of Mach number by performing computations for $E = 0.5$, 0.7, and 0.9 at $Pr = 0.715$. The range of β considered was 0 (flat plate) through 20, where it becomes clear that no new trends will develop. Table 1 and Fig. 1 present the results for r where the effects of Pr and E are clearly exhibited. A definite trend of r towards Pr with increasing β can also be identified. Since it is unlikely that flows with $\beta > 20$ will ever be encountered in engineering practice the computations were not carried to higher values of β in order to establish the asymptotic value by numerical solution. Instead there follows an analytical development of bounds on r and its asymptotic behavior.

From Refs. 4 or 5 we have the governing equations

$$(Cf'')' + ff'' = \beta(f'^2 - g) \tag{1}$$

$$(Cg'/Pr)' + fg' = 2E(Pr^{-1} - 1)(Cf'f'')' \tag{2}$$

subject to the boundary conditions

$$\eta = 0: \ f = f' = g' = 0 \tag{3a}$$

$$\eta \rightarrow \infty: \ f' \rightarrow 1.0, g \rightarrow 1.0 \tag{3b}$$

Formal integration of Eq. (2) subject to the boundary conditions results in

$$1 - g_0 = 2E(1 - Pr) \int_0^\infty (C\phi)^{-1} \int_0^\eta (Cf'f'')'\phi d\bar{\eta} d\eta \tag{4}$$

where

$$\phi = \exp\left[\int_0^\eta (Prf/C) d\bar{\eta}\right] \geq 1 \tag{5}$$

The magnitude of the integral on the right hand side of Eq. (4)

Table 1 Recovery factors for $\omega = 0.5$

	Pr = 0.6	Pr = 0.715			Pr = 0.9
β	$E = 0.9$	$E = 0.5$	$E = 0.7$	$E = 0.9$	$E = 0.9$
0.00	0.7515	0.8387	0.8356	0.8287	0.9424
0.25	0.7329	0.8254	0.8218	0.8147	0.9373
0.50	0.7204	0.8162	0.8126	0.8054	0.9339
1.00	0.7040	0.8039	0.8003	0.7933	0.9294
2.00	0.6856	0.7897	0.7863	0.7795	0.9242
3.00	0.6749	0.7812	0.7780	0.7714	0.9212
4.00	0.6676	0.7753	0.7722	0.7659	0.9192
5.00	0.6627	0.7708	0.7680	0.7618	0.9176
6.00	0.6579	0.7673	0.7646	0.7587	0.9164
7.00	0.6545	0.7644	0.7618	0.7561	0.9155
8.00	0.6517	0.7620	0.7595	0.7540	0.9147
9.00	0.6493	0.7599	0.7576	0.7521	0.9140
10.00	0.6472	0.7582	0.7559	0.7506	0.9134
11.00	0.6454	0.7566	0.7544	0.7492	0.9129
12.00	0.6438	0.7551	0.7530	0.7480	0.9124
13.00	0.6423	0.7539	0.7518	0.7469	0.9120
14.00	0.6410	0.7527	0.7508	0.7459	0.9116
15.00	0.6399	0.7517	0.7498	0.7450	0.9113
16.00	0.6388	0.7507	0.7489	0.7442	0.9110
17.00	0.6378	0.7498	0.7480	0.7434	0.9107
18.00	0.6369	0.7490	0.7473	0.7428	0.9104
19.00	0.6360	0.7482	0.7466	0.7421	0.9102
20.00	0.6353	0.7475	0.7459	0.7415	0.9099

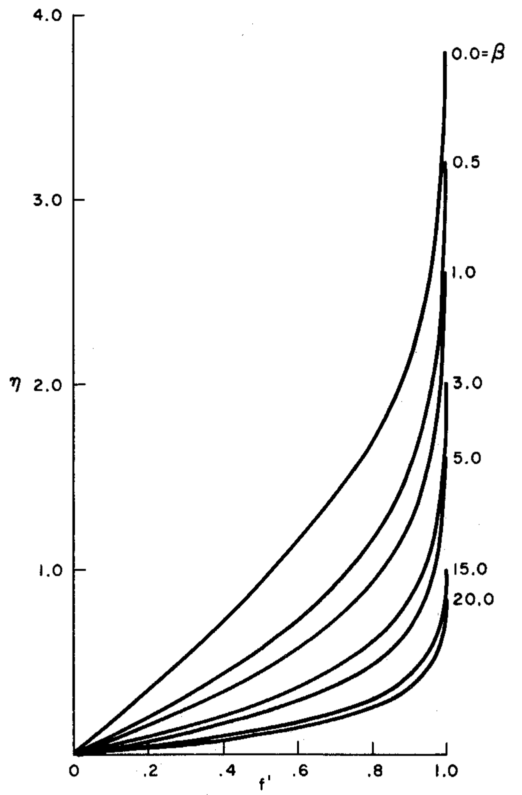

Fig. 2 Effect of accelation on velocity profiles. Pr = 0.715, E = 0.9, ω = 0.5.

is readily estimated if it is noted that

$$\phi^{-1} \leq \exp(-Pr\eta^2/2C_0) \quad (6)$$

with the equality holding only at the inner boundary, $\eta = 0$ Thus

$$\int_0^\infty (C\phi)^{-1} \int_0^\eta (Cf'f'')\phi d\bar{\eta} d\eta > \int_0^\infty (C\phi)^{-1} \times \int_0^\eta (Cf'f'')' d\bar{\eta} d\eta = \int_0^\infty f'f''\phi^{-1} d\eta > \int_0^\infty [f'f'' \exp(-Pr\eta^2/2C_0)] d\eta = \tfrac{1}{2}C_0 \int_0^\infty f'^2 Pr\eta \exp(-Pr\eta^2/2C_0) d\eta > 0$$

Since $Pr < 1$ and $E > 0$ we have from Eq. (4),

$$1 - g_0 > 0 \quad (7)$$

Also, direct integration of the right hand side of Eq. (4) yields

$$1 - g_0 = (1 - Pr)E - 2(1 - Pr)PrE \times \int_0^\infty (C\phi)^{-1} \int_0^\eta f''f'f\phi d\bar{\eta} d\eta \quad (8)$$

In the inner integral all the functions with the exception of f'' are non-negative. In situations where there is a velocity overshoot f'' has to be negative locally, however the integral of f'' must still be positive. Thus the second term on the right hand side of Eq. (8) is positive and

$$1 - g_0 < (1 - Pr)E \quad (9)$$

Recalling now that the recovery factor is defined by the relation $(1 - g_0) = (1 - r)E$, we obtain from Eqs. (7) and (9)

$$Pr < r < 1.0 \quad (10)$$

Thus the recovery factor is less than unity, which it approaches from below as $Pr \to 1.0$. As $\beta \to \infty$ a singular perturbation problem obtains in which the velocity boundary-layer thickness is of the order of $\beta^{-1/2}$ with respect to a total enthalpy layer of order unity. Figure 2 shows the steepening velocity profiles as β increases. Now $ff''\phi \to 0$ as $\eta \to \infty$ so that the integrand of the inner integral in Eq. (8) is finite and tends to zero with increasing η. Also the measure of the interval of integration approaches zero as $\beta \to \infty$. Therefore the inner integral tends to zero as $\beta \to \infty$, and

$$r \to Pr \quad \text{as} \quad \beta \to \infty \quad (11)$$

We have shown analytically that for all finite β the recovery factor is less than unity and that it tends to the Prandtl number as $\beta \to \infty$. Our numerical data confirms these conclusions. The conjectures of Dewey and Gross[1] and Back[2] are therefore incorrect.

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Decay of a Boundary-Layer Induced Shock

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Introduction

The large Reynolds number flow over a flat plate is conceptually well established. The outer, inviscid flow, behaves as though it were displaced by a body equivalent to the displacement thickness of the viscous boundary layer. This effective body generates a shock wave in the inviscid flow and the shock is attenuated by expansion waves which ultimately reduce its strength to zero. A solution for the asymptotic decay of the shock has application in the gas dynamic laser[1] (GDL) where gas density inhomogeneities may cause distortion of the laser beam.

Considering only boundary layers whose displacement thickness (δ^*) increases algebraically with distance (x)

$$\delta^* = Ax^N \quad (0 \leq N \leq 1) \quad (1)$$
$$(A = \text{constant})$$

the strong shock solution may be generated using the hypersonic similarity solutions of Sedov.[2] This, of course, requires a priori knowledge of the boundary-layer growth which is

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